Exercise 63

Find expressions for the first five derivatives of $f(x) = x^2e^x$. Do you see a pattern in these expressions? Guess a formula for $f^{(n)}(x)$ and prove it using mathematical induction.

Solution

Use the product rule to find the first five derivatives of $f(x) = x^2e^x$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(x^2e^x) \\
&= 2xe^x + x^2e^x \\
&= (2x + x^2)e^x \\
f''(x) &= \frac{d}{dx}[(2x + x^2)e^x] \\
&= (2 + 2x)e^x + (2x + x^2)e^x \\
&= (2 + 4x + x^2)e^x \\
f'''(x) &= \frac{d}{dx}[(2 + 4x + x^2)e^x] \\
&= (4 + 2x)e^x + (2 + 4x + x^2)e^x \\
&= (6 + 6x + x^2)e^x \\
f^{(4)}(x) &= \frac{d}{dx}[(6 + 6x + x^2)e^x] \\
&= (6 + 2x)e^x + (6 + 6x + x^2)e^x \\
&= (12 + 8x + x^2)e^x \\
f^{(5)}(x) &= \frac{d}{dx}[(12 + 8x + x^2)e^x] \\
&= (8 + 2x)e^x + (12 + 8x + x^2)e^x \\
&= (20 + 10x + x^2)e^x
\end{aligned}$$

Recognizing the pattern, a formula for the nth derivative can be written.

$$f^{(n)}(x) = [n(n-1) + 2nx + x^2]e^x$$

To prove this, use mathematical induction. Start by checking the base case: Plug in $n = 1$.

$$f^{(1)}(x) = [1(0) + 2(1)x + x^2]e^x = (2x + x^2)e^x$$

This is the correct formula for the first derivative. Now assume that

$$f^{(k)}(x) = [k(k-1) + 2kx + x^2]e^x$$

is true for some integer k. The aim is to show that

$$f^{(k+1)}(x) = [(k+1)(k) + 2(k+1)x + x^2]e^x.$$

Differentiate $f^{(k)}(x)$ with respect to x in order to get $f^{(k+1)}(x)$.

$$\begin{aligned}
\frac{d}{dx}[f^{(k)}(x)] &= f^{(k+1)}(x) \\
&= \frac{d}{dx}\{[k(k-1) + 2kx + x^2]e^x\} \\
&= (2k + 2x)e^x + [k(k-1) + 2kx + x^2]e^x \\
&= [k(k+1) + 2(k+1)x + x^2]e^x
\end{aligned}$$

Therefore, by mathematical induction,

$$f^{(n)}(x) = [n(n-1) + 2nx + x^2]e^x.$$